

Shape invariant potential formalism for photon-added coherent state construction

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An algebro-operator approach, called shape invariant potential method, of constructing generalized coherent states for photon-added particle system is presented. Illustration is given on Pöschl-Teller potential.

Introduction

Coherent states (CS) play an important role in many fields of quantum mechanics since his early days. These states were first introduced by Schrödinger [34] since 1926 for the harmonic oscillator. Then followed decades of intensive works in order to extend the CS concept to other types of exactly solvable systems [5, 6, 17, 20, 28]. It was shown in 1980's that a large class of these solvable potentials are characterized by a single property, i.e., a discrete reparametrization invariance, called shape-invariance [11, 14, 16, 23], introduced in the framework of the supersymmetric quantum mechanics (SUSY QM) [10, 21]. It was then shown that shape invariant potentials (SIP) [11, 23] have an underlying algebraic structure and the associated Lie algebras were identified [2, 7]. Using this algebraic structure, a general definition of coherent states for shape invariant potentials were introduced by different authors [2, 15].

In 1980's, a new class of nonclassical states, known as photon-added coherent states (PACS), was introduced by Agarwal and Tara [1]. These states which are intermediate states between CS and Fock states are constructed by repeated application of the creation operator on an ordinary CS. The PACS have known a great interest, as shown the different extensions [22, 24, 29, 30, 35] and applications of the concept in various field of physics [8, 12].

In a recent work [36], we constructed photon-added CS for SIP and investigated different cases following the Infeld-Hull [21] classification.

In this contribution paper, we aim at providing a rigorous mathematical formulation of the CS and their photon-added counterparts for SIP. We apply this formalism to Pöschl-Teller potentials of great importance in atomic physics. The diagonal P -representation of the density operator ρ is elaborated with thermal expectation values. This computation gives value on the use of Meijer G-functions. Novel results are obtained and discussed.

The paper is organized as follows. In section 2, we review the concepts of SUSY QM factorization, give the algebraic formulation of shape invariance condition, and define the generalized shape-invariant potential coherent states (SIPCS). In section 3, we construct the photon-added shape-invariant potentials coherent states (PA-SIPCS) by successive applications of the raising operator on the SIP-CS. We calculate the inner product of two different PA-SIPCS in order to show that the obtained states are not mutually orthogonal. In contrast, we prove that these states are normalized. The resolution of unity is checked. Finally, we study the thermal statistical properties of the PA-SIPCS in terms of the Mandels Q-parameter.

In section 4, Pöschl-Teller potentials are investigated as illustration. We end, in section 5, with some concluding remarks.

1 Mathematical formulation of SUSYQM: integrability condition and coherent state construction

In this section, we introduce the SUSY QM factorization method [19] (and references therein), give the integrability condition, known as shape invariance condition, and define the associated generalized coherent states.

Let $\mathcal{H} = L^2(]a, b[, dx)$ be the Hilbert space with the inner product defined by :

$$\langle u, v \rangle := \int_a^b \bar{u}(x)v(x)dx, \quad \forall u, v \in \mathcal{H}, \quad (1.1)$$

where \bar{u} is the complex conjugate of u . Consider on \mathcal{H} the one-dimensional bound-state Hamiltonian ($\hbar = 2m = 1$)

$$H = -\frac{d^2}{dx^2} + V(x), \quad x \in]a, b[\subset \mathbb{R} \quad (1.2)$$

with the domain

$$\mathcal{D}(H) = \{u \in \mathcal{H}, \quad -u'' + Vu \in \mathcal{H}\}, \quad (1.3)$$

where $V(x)$ is a real continuous function on $]a, b[$. Let us denote E_n and Ψ_n the eigenvalues and eigenfunctions of H , respectively. Let the first-order differential operator A be defined by:

$$A = \frac{d}{dx} + W(x), \text{ with the domain } \mathcal{D}(A) = \{u \in \mathcal{H}, \quad u' + Wu \in \mathcal{H}\}, \quad (1.4)$$

$W(x) = -\frac{d}{dx}[\ln(\Psi_0)]$ is a real continuous functions on $]a, b[$. The adjoint operator A^\dagger of A is defined on [37]:

$$\mathcal{D}(A^\dagger) = \{u \in \mathcal{H} \mid \exists \tilde{v} \in \mathcal{H} : \langle Au, v \rangle = \langle u, \tilde{v} \rangle \quad \forall u \in \mathcal{D}(A)\}, \quad A^\dagger v = \tilde{v}. \quad (1.5)$$

We infer $\mathcal{D}(A)$ dense in \mathcal{H} since $H^{1,2}(]a, b[, \rho(x)dx)$ is dense in \mathcal{H} and $H^{1,2}(]a, b[, \rho(x)dx) \subset \mathcal{D}(H)$, where $H^{m,n}(\Omega)$ is the Sobolev spaces of indices (m, n) . We assume that the operator A is closed in \mathcal{H} . The explicit expression of A^\dagger is given through the following theorem.

Theorem 1.1 *Suppose the following boundary condition:*

$$u(x)v(x)\Big|_a^b = 0, \quad \forall u \in \mathcal{D}(A) \text{ and } v \in \mathcal{D}(A^\dagger), \quad (1.6)$$

is verified. Then the operator A^\dagger can be written as

$$A^\dagger = \left[-\frac{d}{dx} + W(x) \right]. \quad (1.7)$$

Proof: The proof follows as

$$\begin{aligned} \langle A\bar{u}, v \rangle &\equiv \int_a^b [\bar{u}'(x) + W(x)\bar{u}(x)] v(x)dx \\ &= \bar{u}(x)v(x)\Big|_a^b + \int_a^b \bar{u}(x) \left[-\frac{d}{dx} + W(x) \right] v(x)dx \\ &= \langle \bar{u}, A^\dagger v \rangle \quad \text{for any } u \in \mathcal{D}(A), \quad v \in \mathcal{D}(A^\dagger). \end{aligned}$$

■

Let H_1 and H_2 be the product operators $A^\dagger A$ and $A A^\dagger$, respectively, with the corresponding domains

$$\begin{aligned}\mathcal{D}(H_1) &= \{u \in \mathcal{D}(A), v = Au \in \mathcal{D}(A^\dagger) \text{ and } A^\dagger v \in \mathcal{H}\}, \\ \mathcal{D}(H_2) &= \{u \in \mathcal{D}(A^\dagger), v = A^\dagger u \in \mathcal{D}(A) \text{ and } Av \in \mathcal{H}\}.\end{aligned}\quad (1.8)$$

Remark that

$$H^{1,2}([a, b[, dx) \subset \mathcal{D}(A) \subset \mathcal{D}(A^\dagger).$$

Then

$$\overline{\mathcal{D}(H_1), \mathcal{D}(H_2)} \supset H^{2,2}([a, b[, dx).$$

We infer then that $\mathcal{D}(H_1)$ and $\mathcal{D}(H_2)$ are dense in \mathcal{H} . The following theorem gives additional conditions on W so that the operator H factorizes in terms of A and A^\dagger .

Theorem 1.2 *Suppose that the function W verifies the Riccati type equation:*

$$V - E_0 = W^2 - W'. \quad (1.9)$$

Then the operators $H_{1,2}$ are self-adjoint, and:

$$H_1 = A^\dagger A = H - E_0 = -\frac{d^2}{dx^2} + W^2 - W', \quad H_2 = A A^\dagger = -\frac{d^2}{dx^2} + W^2 + W'. \quad (1.10)$$

Proof The operators $A^\dagger A$ and $A A^\dagger$ are self-adjoint since A and A^\dagger are mutually adjoint and A is closed with $\mathcal{D}(A)$ dense in \mathcal{H} . From the definitions (1.4) et (1.7) of the differential operators A and A^\dagger , we have the following products

$$A^\dagger A = -\frac{d^2}{dx^2} + (W^2 - W'), \quad A A^\dagger = -\frac{d^2}{dx^2} + (W^2 + W').$$

The equation (1.9) are readily deduced from the above relations and (1.10). ■

We can rewrite the operators $H_{1,2}$ as:

$$H_{1,2} = -\frac{d^2}{dx^2} + V_{1,2}, \quad \text{where } V_{1,2} = W^2 \mp W'. \quad (1.11)$$

In SUSY QM terminology, $H_{1,2}$ are called SUSY partner Hamiltonians; $V_{1,2}$ are called SUSY partner potentials, and the function W is called the superpotential.

Now let us establish some results showing that the eigenvalues of partner Hamiltonians are positive definite ($E_n^{1,2} \geq 0$) and isospectral, i.e, they have almost the same energy eigenvalues, except for the ground state energy of H_1 [10].

Proposition 1.3 *The eigenvalues of H_1 and H_2 are non negative*

$$E_n^{(1)} \geq 0, \quad E_n^{(2)} \geq 0.$$

Proof: Let $E_n^{(1)}$ be an eigenvalue of H_1 corresponding to the eigenfunction $\Psi_n^{(1)}$. In Dirac notation, this reads as $H_1|\Psi_n^{(1)}\rangle = E_n^{(1)}|\Psi_n^{(1)}\rangle$. Then $\langle \Psi_n^{(1)} | A^\dagger A | \Psi_n^{(1)} \rangle = E_n^{(1)} \langle \Psi_n^{(1)} | \Psi_n^{(1)} \rangle$, i.e, $\|A|\Psi_n^{(1)}\rangle\|^2 = E_n^{(1)}\|\Psi_n^{(1)}\|^2$. Therefore $E_n^{(1)} \geq 0$, since $\|A|\Psi_n^{(1)}\rangle\|^2 \geq 0$ and $\|\Psi_n^{(1)}\|^2 \geq 0$. Similarly, one can show that $E_n^{(2)} \geq 0$. ■

Proposition 1.4 *Let $|\Psi_n^{(1)}\rangle$ and $|\Psi_n^{(2)}\rangle$ be the normalized eigenstates of H_1 and H_2 associated to the eigenvalues $E_n^{(1)}$ and $E_n^{(2)}$, respectively. Then*

$$A|\Psi_n^{(1)}\rangle = 0 \iff E_n^{(1)} = 0, \quad A^\dagger|\Psi_n^{(2)}\rangle = 0 \iff E_n^{(2)} = 0.$$

Proof:

$$\begin{aligned}
A|\Psi_n^{(1)}\rangle = 0 &\iff ||A|\Psi_n^{(1)}\rangle||^2 = 0 \\
&\iff \langle \Psi_n^{(1)} | A^\dagger A |\Psi_n^{(1)}\rangle = 0 \\
&\iff E_n^{(1)} \langle \Psi_n^{(1)} | \Psi_n^{(1)}\rangle = 0 \\
&\iff E_n^{(1)} = 0.
\end{aligned}$$

By analogy, one can show that $A^\dagger |\Psi_n^{(2)}\rangle = 0 \iff E_n^{(1)} = 0$. ■

As a consequence of this proposition $E_0^{(1)} = 0$, since $A\Psi_0^{(1)} = A\Psi_0 = 0$.

Proposition 1.5 *If H_1 admits a normalized eigenstate $|\Psi_0^{(1)}\rangle$ so that $E_0^{(1)} = 0$, then H_2 does not admit a normalized eigenstate $|\Psi_0^{(2)}\rangle$ corresponding to the eigenvalue $E_0^{(2)} = 0$.*

Proof: If $E_0^{(1)} = 0$, then from the proposition 1.4, $A\Psi_0^{(1)} = 0$. We deduce from this that

$$AA^\dagger A\Psi_0^{(1)} = H_2(A\Psi_0^{(1)}) = 0. \quad (1.12)$$

Suppose that there exists a normalizable eigenstate $|\Psi_0^{(2)}\rangle$ of H_2 corresponding to $E_0^{(2)} = 0$. It follows from (1.12) that $|\Psi_0^{(2)}\rangle \propto A\Psi_0^{(1)} = 0$, that is inconsistent. ■

This proposition shows that H_2 cannot possess a normalized state $\Psi_0^{(2)}$ corresponding to the eigenvalues $E_0^{(2)} = 0$, since $E_0^{(1)} = 0$, that means $E_0^{(2)} \neq 0$.

Proposition 1.6 *Let $|\Psi_n^{(1)}\rangle$ and $|\Psi_n^{(2)}\rangle$ be normalized eigenstates of H_1 and H_2 , respectively, such that $A|\Psi_n^{(1)}\rangle \neq 0$, $A|\Psi_n^{(2)}\rangle \neq 0$ and the corresponding eigenvalues are, respectively, $E_n^{(1)} \neq 0$ and $E_n^{(2)} \neq 0$. Then $E_n^{(1)}$ is also an eigenvalue of H_2 associated to the eigenstate*

$$|\Psi_{n-1}^{(2)}\rangle = (E_n^{(1)})^{-1/2} A|\Psi_n^{(1)}\rangle;$$

$E_n^{(2)}$ is also an eigenvalue of H_1 associated to the eigenstate

$$|\Psi_{n+1}^{(2)}\rangle = (E_n^{(1)})^{-1/2} A^\dagger |\Psi_n^{(1)}\rangle.$$

Proof: We have $H_1|\Psi_n^{(1)}\rangle = E_n^{(1)}|\Psi_n^{(1)}\rangle$. From this, we deduce that $AA^\dagger(A|\Psi_n^{(1)}\rangle) = E_n^{(1)}(A|\Psi_n^{(1)}\rangle)$, or

$$H_2(A|\Psi_n^{(1)}\rangle) = E_n^{(1)}(A|\Psi_n^{(1)}\rangle), \quad (1.13)$$

i.e, $A|\Psi_n^{(1)}\rangle$ is an eigenstate of H_2 associated to the eigenvalue $E_n^{(1)}$. Similarly, $H_2|\Psi_n^{(2)}\rangle = E_n^{(2)}|\Psi_n^{(2)}\rangle$ implies $A^\dagger A(A^\dagger |\Psi_n^{(2)}\rangle) = E_n^{(2)}(A^\dagger |\Psi_n^{(2)}\rangle)$, i.e,

$$H_1(A^\dagger |\Psi_n^{(2)}\rangle) = E_n^{(2)}(A^\dagger |\Psi_n^{(2)}\rangle). \quad (1.14)$$

This means that $A^\dagger |\Psi_n^{(2)}\rangle$ is an eigenstate of H_1 with the eigenvalue $E_n^{(2)}$. The eigenvalues of H_1 being non degenerate (since we consider only bound state of H_1), it follows that there exists a unique normalized eigenstate $|\Psi_k^{(1)}\rangle$ of H_1 , up to a multiplicative constant, corresponding to an eigenvalue $E_k^{(1)}$ such that $|\Psi_k^{(1)}\rangle = cA^\dagger |\Psi_n^{(2)}\rangle$. The normalization constant c is given by $c = (E_n^{(2)})^{-1/2}$. We have

$$|\Psi_k^{(1)}\rangle = (E_n^{(2)})^{-1/2} A^\dagger |\Psi_n^{(2)}\rangle. \quad (1.15)$$

It follows from (1.15) that

$$\begin{aligned}
H_1|\Psi_k^{(1)}\rangle &= (E_n^{(2)})^{-1/2} H_1(A^\dagger |\Psi_n^{(2)}\rangle) \\
&= (E_n^{(2)})^{-1/2} E_n^{(2)}(A^\dagger |\Psi_n^{(2)}\rangle) && \text{from (1.14)} \\
&= E_n^{(2)}|\Psi_k^{(1)}\rangle && \text{from (1.15)}.
\end{aligned}$$

Then $H_1|\Psi_k^{(1)}\rangle = E_k^{(1)}|\Psi_k^{(1)}\rangle = E_n^{(2)}|\Psi_k^{(1)}\rangle$. It follows from this that $E_k^{(1)} = E_n^{(2)}$. Since $E_0^{(1)} \neq E_0^{(2)}$ ($E_0^{(1)} = 0$ and $E_0^{(2)} \neq 0$), a simplest solution of the index equation is $k = n + 1$. Hence

$$\begin{cases} E_{n+1}^{(1)} &= E_n^{(2)} \\ |\Psi_{n+1}^{(1)}\rangle &= (E_n^{(2)})^{-1/2}(A^\dagger|\Psi_n^{(2)}\rangle). \end{cases} \quad (1.16)$$

One can similarly show from (1.13) that

$$\begin{cases} E_n^{(2)} &= E_{n+1}^{(1)} \\ |\Psi_n^{(2)}\rangle &= (E_{n+1}^{(1)})^{-1/2}(A|\Psi_{n+1}^{(1)}\rangle). \end{cases} \quad (1.17)$$

■

It follows from these propositions that the eigenvalues of H_1 and H_2 are positive definite ($E_n^{1,2} \geq 0$), and the partner Hamiltonians are isospectral, i.e., they have almost the same energy eigenvalues, except for the ground state energy of H_1 which is missing in the spectrum of H_2 . The spectra are linked as [10]:

$$\begin{aligned} E_n^{(2)} &= E_{n+1}^{(1)}, \quad E_0^{(1)} = 0, \quad n = 0, 1, 2, \dots, \\ \Psi_n^{(2)} &= \left[E_{n+1}^{(1)}\right]^{(-1/2)} A \Psi_{n+1}^{(1)}, \\ \Psi_{n+1}^{(1)} &= \left[E_n^{(2)}\right]^{(-1/2)} A^\dagger \Psi_n^{(2)}. \end{aligned} \quad (1.18)$$

Hence, if the eigenvalues and eigenfunctions of one of the partner, say H_1 , are known, one can immediately derive the eigenvalues and eigenfunctions of H_2 .

However, the above relations (1.18) only give the relationship between the eigenvalues and eigenfunctions of the two partner Hamiltonians, but do not allow to determine their spectra. A condition of an exact solvability is known as the shape invariance condition; that is, the pair of SUSY partner potentials $V_{1,2}$ are similar in shape and differ only in the parameters that appear in them. Gendenshtein states the shape invariance condition as [10, 16]

$$V_2(x; a_1) = V_1(x; a_2) + \mathcal{R}(a_1), \quad (1.19)$$

where a_1 is a set of parameters and a_2 is a function of a_1 , ($a_2 = f(a_1)$), and $\mathcal{R}(a_1)$ is the non-vanishing remainder independent of x . In such a case, the eigenvalues and the eigenfunctions of H_1 can explicitly be deduced [16]. If this Hamiltonian H_1 has p ($p \geq 1$) bound states with eigenvalues $E_n^{(1)}$, and eigenfunctions $\Psi_n^{(1)}$ with $0 \leq n \leq p-1$, the starting point of constructing the spectra is to generate a hierarchy of $(p-1)$ Hamiltonians H_2, \dots, H_p such that the m 'th member of the hierarchy (H_m) has the same spectrum as H_1 except that the first $m-1$ eigenvalues of H_1 are missing in the spectrum of H_m [10]. In order m , ($m = 2, 3, \dots, p$), we have partner Hamiltonians

$$\begin{aligned} H_m(x; a_1) &= A_m^\dagger(x; a_1)A_m(x; a_1) + E_0^{(m)} = -\frac{d^2}{dx^2} + V_m(x; a_1), \\ H_{m+1}(x; a_1) &= A_m(x; a_1)A_m^\dagger(x; a_1) + E_0^{(m)} = -\frac{d^2}{dx^2} + V_{m+1}(x; a_1), \end{aligned}$$

the spectra of which are related as

$$E_n^{(m+1)} = E_{n+1}^{(m)}, \quad \Psi_n^{(m+1)} = (E_{n+1}^{(m)} - E_0^{(m)})A_m\Psi_{n+1}^{(m)}.$$

In terms of the spectrum of H_1 we have

$$\begin{aligned} E_n^{(m)} &= E_{n+1}^{(m-1)} = E_{n+2}^{(m-2)} = \dots = E_{n+m-1}^{(1)} \\ \Psi_n^{(m)} &= (E_{n+m-1}^{(1)} - E_{m-2}^{(1)})^{-1/2} \dots (E_{n+m-1}^{(1)} - E_0^{(1)})^{-1/2} A_{m+1} \dots A_1 \Psi_{n+m-1}^{(1)}(x; a_1). \end{aligned} \quad (1.20)$$

Theorem 1.7 *The eigenvalues of H_1 are given by [10, 16]*

$$E_n^{(1)} = \sum_{k=1}^n \mathcal{R}(a_k). \quad (1.21)$$

Proof: Consider the partner Hamiltonians H_m and H_{m+1} of the hierarchy of Hamiltonians constructed from H_1 . If the partner potentials are shape invariant, we can write

$$\begin{aligned} V_{m+1}(x; a_1) &= V_m(x; a_2) + \mathcal{R}(a_1) \\ &= V_{m-1}(x; a_3) + \mathcal{R}(a_2) + \mathcal{R}(a_1) \\ &= V_{m-2}(x; a_4) + \mathcal{R}(a_3) + \mathcal{R}(a_2) + \mathcal{R}(a_1) \\ &\vdots \\ &= V_2(x; a_m) + \mathcal{R}(a_{m-1}) + \mathcal{R}(a_{m-2}) + \cdots + \mathcal{R}(a_1) \\ &= V_1(x; a_{m+1}) + \sum_{k=1}^m \mathcal{R}(a_k). \end{aligned}$$

It follows from the above that $H_m(x; a_1) = H_1(x; a_m) + \sum_{k=1}^{m-1} \mathcal{R}(a_k)$. Hence $E_0^{(m)} = \sum_{k=1}^{m-1} \mathcal{R}(a_k)$. From equation (1.18), $E_0^{(m)} = E_{m-1}^{(1)}$. Then $E_{m-1}^{(1)} = \sum_{k=1}^{m-1} \mathcal{R}(a_k)$, i.e., $E_n^{(1)} = \sum_{k=1}^n \mathcal{R}(a_k)$. ■

Theorem 1.8 *The normalized eigenfunctions of H_1 are given by [11]*

$$\Psi_n(x; a_1) = \left\{ \prod_{k=1}^n \left(\sum_{p=1}^k \mathcal{R}(a_p) \right) \right\}^{-1/2} A^\dagger(x; a_1) \cdots A^\dagger(x; a_n) \Psi_0^{(1)}(x; a_{n+1}). \quad (1.22)$$

Proof: From the shape invariance condition (1.19), we deduce the following relation between the eigenfunctions of the partner Hamiltonians H_1 and H_2

$$\Psi_n^{(2)}(x; a_1) = \Psi_n^{(1)}(x; a_2) \quad (1.23)$$

We know from (1.17) that

$$\begin{aligned} \Psi_{n+1}^{(1)}(x; a_1) &= (E_n^{(2)})^{-1/2} A^\dagger(x; a_1) \Psi_n^{(2)}(x; a_1) \\ &= (E_n^{(2)})^{-1/2} A^\dagger(x; a_1) \Psi_n^{(1)}(x; a_2) \quad \text{from (1.23)} \\ &= (E_n^{(2)})^{-1/2} (E_{n-1}^{(2)})^{-1/2} A^\dagger(x; a_1) A^\dagger(x; a_2) \Psi_{n-1}^{(2)}(x; a_3) \\ &= \vdots \\ &= (E_n^{(2)})^{-1/2} \cdots (E_0^{(2)})^{-1/2} A^\dagger(x; a_1) \cdots A^\dagger(x; a_{n+1}) \Psi_0^{(1)}(x; a_{n+2}). \end{aligned}$$

It deduces from above equations that

$$\Psi_n^{(1)}(x; a_1) = \left\{ \prod_{k=1}^n \left(\sum_{p=1}^k \mathcal{R}(a_p) \right) \right\}^{-1/2} A^\dagger(x; a_1) \cdots A^\dagger(x; a_n) \Psi_0^{(1)}(x; a_{n+1}). \quad (1.24)$$

The shape invariance condition (1.19) can be rewritten in terms of the factorization operators defined in equations (1.4)-(1.7),

$$A(a_1)A^\dagger(a_1) = A^\dagger(a_2)A(a_2) + \mathcal{R}(a_1), \quad (1.25)$$

where a_2 is a function of a_1 . Here, we consider only the translation class of shape invariance potentials, that is the case where the parameters a_1 and a_2 are related as $a_2 = a_1 + \eta$ [11] and the potentials are known in closed form. The scaling class [23] is not treated here since the potentials, in this case, can only be written as Taylor expansion.

Introducing a reparametrization operator T_η defined as

$$T_\eta : \mathcal{H} \longrightarrow \mathcal{H} \quad T_\eta \Phi(x; a_1) := \phi(x; a_1 + \eta) = \Phi(x; a_2) \quad (1.26)$$

that replaces a_1 with a_2 in a given operator [7]

$$T_\eta \mathcal{O}(a_1) T_\eta^{-1} = \mathcal{O}(a_1 + \eta) := \mathcal{O}(a_2), \quad (1.27)$$

and the operators

$$B_-, B_+ : \mathcal{H} \longrightarrow \mathcal{H} \quad B_+ = A^\dagger(a_1) T_\eta, \quad B_- = T_\eta^\dagger A(a_1), \quad (1.28)$$

with the domains

$$\mathcal{D}(B_-) = \{u \in \mathcal{H}, \quad v = u' + W u \in \mathcal{H} \quad \text{and} \quad T_\eta^\dagger v \in \mathcal{H}\} \quad (1.29)$$

$$\mathcal{D}(B_+) = \left\{ u \in \mathcal{H}, \quad v = T_\eta u \in \mathcal{H} \quad \text{and} \quad -v' + W v \in \mathcal{H} \right\}. \quad (1.30)$$

The Hamiltonian factorizes in terms of the new operators as follow:

$$H - E_0 = H_1 = A^\dagger(a_1) A(a_1) = B_+ B_- \quad (1.31)$$

where

$$[B_-, B_+] = \mathcal{R}(a_0) \quad , \quad B_- |\Psi_0\rangle = 0. \quad (1.32)$$

The states $B_+^n |\Psi_0\rangle$ are eigenfunctions of H with eigenvalues E_n , ie,

$$H(B_+^n |\Psi_0\rangle) = \underbrace{\left[\sum_{k=1}^n \mathcal{R}(a_k) \right]}_{E_n} B_+^n |\Psi_0\rangle \quad (1.33)$$

B_\pm act as raising and lowering operators:

$$B_+ |\Psi_n\rangle = \sqrt{E_{n+1}} |\Psi_{n+1}\rangle \quad , \quad B_- |\Psi_n\rangle = \sqrt{\mathcal{R}(a_0) + E_{n-1}} |\Psi_{n-1}\rangle. \quad (1.34)$$

To define shape-invariant potential coherent states, Balantekin *et al* [2] introduced the right inverse of B_- as: $B_- B_-^{-1} = \mathbb{1}$ and the left inverse H^{-1} of H such that: $H^{-1} B_+ = B_-^{-1}$. The SIPCS defined by

$$|z\rangle = \sum_{n=0}^n (z B_-^{-1})^n |\Psi_0\rangle \quad (1.35)$$

are eigenstates of the lowering operator B_- :

$$B_- |z\rangle = z |z\rangle. \quad (1.36)$$

A generalization of the SIPCS (1.35) was done as [2]:

$$|z; a_j\rangle = \sum_{n=0}^{\infty} \{z Z_j B_-^{-1}\}^n |\Psi_0\rangle, \quad z, Z_j \in \mathbb{C} \quad (1.37)$$

where $Z_j \equiv Z_{(a_j)} \equiv Z(a_1, a_2, \dots)$. Observing that $B_-^{-1} Z_j = Z_{j+1} B_-^{-1}$ and from

$$Z_{j-1} = T^\dagger(a_1) Z_j T(a_1) \quad (1.38)$$

one can readily show that

$$(zZ_j B_-^{-1})^n = z^n \prod_{k=0}^{n-1} Z_{j+k} B_-^{-n} . \quad (1.39)$$

Using (1.39), one can straightforwardly deduce that (1.37) are eigenstates of B_- :

$$B_- |z; a_j\rangle = z Z_{j-1} |z; a_j\rangle . \quad (1.40)$$

Observing that

$$B_-^{-n} |\Psi_0\rangle = C_n |\psi_n\rangle , \quad C_n = \left[\prod_{k=1}^n \left(\sum_{s=k}^n \mathcal{R}(a_s) \right) \right]^{-1/2} \quad (1.41)$$

and using (1.41), the normalized form of the CS (1.37) can be obtained as:

$$|z; a_r\rangle = \mathcal{N}(|z|^2; a_r) \sum_{n=0}^{\infty} \frac{z^n}{h_n(a_r)} |\Psi_n\rangle , \quad (1.42)$$

where we used the shorthand notation $a_r \equiv [\mathcal{R}(a_1), \mathcal{R}(a_2), \dots, \mathcal{R}(a_n); a_j, a_{j+1}, \dots, a_{j+n-1}]$. The expansion coefficient $h_n(a_r)$ and the normalization constant $\mathcal{N}(|z|^2; a_r)$ are:

$$h_n(a_r) = \frac{\sqrt{\prod_{k=1}^n \left(\sum_{s=k}^n \mathcal{R}(a_s) \right)}}{\prod_{k=0}^{n-1} Z_{j+k}} \quad \text{for } n \geq 1, \quad h_0(a_r) = 1, \quad \mathcal{N}(x; a_r) = \left[\sum_{n=0}^{\infty} \frac{x^n}{|h_n(a_r)|^2} \right]^{-1/2} . \quad (1.43)$$

It is shown [2] that these states (1.37) fulfill the standard properties of label continuity, overcompleteness, temporal stability and action identity.

2 Construction of photon-added coherent states for shape invariant systems

In this section, a construction of PA-SIPCS [36], and their physical and mathematical properties are presented.

2.1 Definition of the PA-SIPCS

Let \mathfrak{H}_m be the Hilbert subspace of \mathfrak{H} defined as follows:

$$\mathfrak{H}_m := \text{span} \{ |\Psi_{n+m}\rangle \}_{n,m \geq 0} . \quad (2.1)$$

By successive application of the raising operator B_+ on the generalized SIPCS (1.36), we can obtain photon-added shape-invariant potential CS (PA-SIPCS) denoted by $|z; a_r\rangle_m$:

$$|z; a_r\rangle_m := (B_+^m) |z; a_r\rangle \quad (2.2)$$

where m is a positive integer standing for the number of added quanta or photons.

It is worth mentioning that the first m eigenstates $|\Psi_n\rangle$, $n = 0, 1, \dots, m-1$ are absent from the wave-function $|z; a_r\rangle_m \in \mathfrak{H}_m$. Therefore, from the orthonormality relation satisfied by the states $|\Psi_n\rangle$, the overcompleteness relation fulfilled by the identity operator on \mathfrak{H}_m , denoted by $\mathbb{1}_{\mathfrak{H}_m}$, is to be written as [29, 35]

$$\mathbb{1}_{\mathfrak{H}_m} = \sum_{n=m}^{\infty} |\Psi_n\rangle \langle \Psi_n| = \sum_{n=0}^{\infty} |\Psi_{n+m}\rangle \langle \Psi_{n+m}| . \quad (2.3)$$

Here, $\mathbb{1}_{\mathfrak{H}_m}$ is only required to be a bounded positive operator with a densely defined inverse [4].

From (1.27) and using the relations $B_+ \mathcal{R}(a_{n-1}) = \mathcal{R}(a_n) B_+$ and $B_+ |\Psi_n\rangle = \sqrt{E_{n+1}} |\Psi_{n+1}\rangle$, we obtain the PA-SIPCS as:

$$|z; a_r\rangle_m = \mathcal{N}_m(|z|^2; a_r) \sum_{n=0}^{\infty} \frac{z^n}{K_n^m(a_r)} |\Psi_{n+m}\rangle \quad (2.4)$$

where the expansion coefficient takes the form:

$$K_n^m(a_r) = \frac{\left[\prod_{k=m+1}^{n+m} \left(\sum_{s=k}^{n+m} \mathcal{R}(a_s) \right) \right]^{1/2}}{\left[\prod_{k=m}^{n+m-1} Z_{j+k} \right] \cdot \left[\prod_{k=1}^m \left(\sum_{s=k}^{n+m} \mathcal{R}(a_s) \right) \right]^{1/2}}, \quad (2.5)$$

and the normalization constant $\mathcal{N}_m(|z|^2; a_r)$ is given by:

$$\mathcal{N}_m(|z|^2; a_r) = \left(\sum_{n=0}^{\infty} \frac{|z|^{2n}}{|K_n^m(a_r)|^2} \right)^{-1/2}. \quad (2.6)$$

The inner product of two different PA-SIPCS $|z; a_r\rangle_m$ and $|z'; a_r\rangle_{m'}$

$${}_{m'} \langle z'; a_r | z; a_r \rangle_m = \mathcal{N}_{m'}(|z'|^2; a_r) \mathcal{N}_m(|z|^2; a_r) \sum_{n, n'=0}^{\infty} \frac{z'^{\star n'} z^n}{K_{n'}^{m'\star}(a_r) K_n^m(a_r)} \langle \Psi_{n'+m'} | \Psi_{n+m} \rangle \quad (2.7)$$

does not vanish. Indeed, due to the orthonormality of the eigenstates $|\Psi_n\rangle$, the inner product (2.7) can be rewritten as

$${}_{m'} \langle z'; a_r | z; a_r \rangle_m = \mathcal{N}_{m'}(|z'|^2; a_r) \mathcal{N}_m(|z|^2; a_r) z'^{\star(m-m')} \sum_{n=0}^{\infty} \frac{(z'^{\star} z)^n}{K_{n+m-m'}^{m'\star}(a_r) K_n^m(a_r)}, \quad (2.8)$$

showing that the PA-SIPCS are not mutually orthogonal.

2.2 Label continuity

In the Hilbert space \mathfrak{H} , the PA-SIPCS $|z, a_r\rangle_m$ are labeled by m and z . The label continuity condition can then be stated as:

$$|z - z'| \rightarrow 0 \text{ and } |m - m'| \rightarrow 0 \implies ||z, a_r\rangle_m - |z', a_r\rangle_{m'}|^2 = 2 [1 - \mathcal{R}e({}_{m'} \langle z'; a_r | z; a_r \rangle_m)] \rightarrow 0. \quad (2.9)$$

This is satisfied by the states $|z, a_r\rangle_m$, since from Eqs. (2.6, 2.8), we see that

$$m \rightarrow m' \text{ and } z \rightarrow z' \implies {}_{m'} \langle z'; a_r | z; a_r \rangle_m \rightarrow 1. \quad (2.10)$$

Therefore the PA-SIPCS $|z, a_r\rangle_m$ are continuous in their labels.

2.3 Overcompleteness

We check the realization of the resolution of identity in the Hilbert space (2.1) with the identity operator defined as (2.3):

$$\int_{\mathbb{C}} d^2 z |z; a_r\rangle_m \omega_m(|z|^2; a_r) {}_m \langle z; a_r| = \mathbb{1}_{\mathfrak{H}_m}. \quad (2.11)$$

Inserting the definition (2.4) of the PA-SIPCS $|z; a_r\rangle_m$ into Eq. (2.11) yields, after taking the angular integration of the diagonal matrix elements:

$$\int_0^{\infty} dx x^n \mathcal{W}_m(x; a_r) = |K_n^m(a_r)|^2, \quad \text{with } \mathcal{W}_m(x; a_r) = \pi \mathcal{N}_m^2(x; a_r) \omega_m(x; a_r). \quad (2.12)$$

Therefore, the weight function ω_m is related to the undetermined moment distribution $\mathcal{W}_m(x; a_r)$, which is the solution of the Stieltjes moment problem with the moments given by $|K_n^m(a_r)|^2$. In order to use Mellin transformation, we can rewrite (2.12) as

$$\int_0^\infty dx x^{n+m} g_m(x; a_r) = |K_n^m(a_r)|^2, \quad \text{where} \quad g_m(x; a_r) = \pi \mathcal{N}_m^2(x; a_r) x^{-m} \omega_m(x; a_r). \quad (2.13)$$

By performing the variable change $n + m \rightarrow s - 1$, Eq. (2.13) becomes:

$$\int_0^\infty dx x^{s-1} g_m(x; a_r) = |K_s^m(a_r)|^2. \quad (2.14)$$

Comparing this relation with the Meijer's G-function and the Mellin inversion theorem [26]

$$\int_0^\infty dx x^{s-1} G_{p,q}^{m,n} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_n; a_{n+1}, \dots, a_p \\ b_1, \dots, b_m; b_{m+1}, \dots, b_q \end{matrix} \right. \right) = \frac{1}{\alpha^s} \frac{\prod_{j=1}^m \Gamma(b_j + s) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - s) \prod_{j=n+1}^p \Gamma(a_j + s)}, \quad (2.15)$$

we see that if $|K_s^m(a_r)|^2$ in the above relation can be expressed in terms of Gamma functions, then $g_m(x; a_r)$ can be identified as the Meijer's G-function.

2.4 Thermal statistics

In quantum mechanics, the density matrix, generally denoted by ρ , is an important tool for characterizing the probability distribution on the states of a physical system. For example, it is useful for examining the physical and chemical properties of a system (see [29], [6] and references listed therein). Consider a quantum gas of the system in the thermodynamic equilibrium with a reservoir at temperature T , which satisfies a quantum canonical distribution. The corresponding normalized density operator is given, in the Hilbert space $\mathfrak{H}_m := \text{span} \{ |\Psi_{n+m}\rangle \}_{n,m \geq 0}$, as

$$\rho^{(m)} = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\beta E_n} |\Psi_{n+m}\rangle \langle \Psi_{n+m}|, \quad (2.16)$$

where in the exponential E_n is the eigen-energy, and the partition function Z is taken as the normalization constant.

The diagonal elements of $\rho^{(m)}$, essential for our purpose, also known as the Q -distribution or Husimi's distribution, are derived in the PA-SIPCS basis as

$${}_m \langle z; a_r | \rho^{(m)} | z; a_r \rangle_m = \frac{\mathcal{N}_m^2(|z|^2; m)}{Z} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{|K_n^p(a_r)|^2} e^{-\beta E_n}. \quad (2.17)$$

The normalization of the density operator leads to

$$\text{Tr} \rho^{(m)} = \int_{\mathbb{C}} d^2 z \omega_m(|z|^2; a_r) {}_m \langle z; a_r | \rho^{(m)} | z; a_r \rangle_m = 1. \quad (2.18)$$

The diagonal expansion of the normalized canonical density operator over the PA-SIPCS projector is

$$\rho^{(m)} = \int_{\mathbb{C}} d^2 z \omega_m(|z|^2; a_r) |z; a_r\rangle_m P(|z|^2)_m \langle z; a_r|, \quad (2.19)$$

where the P -distribution function $P(|z|^2)$ satisfying the normalization to unity condition

$$\int_{\mathbb{C}} d^2 z \omega_m(|z|^2; a_r) P(|z|^2) = 1 \quad (2.20)$$

must be determined.

Thus, given an observable \mathcal{O} , one obtains the expectation value, i. e., the thermal average given by

$$\langle \mathcal{O} \rangle_m = \text{Tr}(\rho^{(m)} \mathcal{O}) = \int_{\mathbb{A}} d^2 z \omega_m(|z|^2; a_r) P(|z|^2)_m \langle z; a_r | \mathcal{O} | z; a_r \rangle_m. \quad (2.21)$$

One can check that for a PA-SIPCS (2.4) the expectation values of the operator $N := B_+ B_-$ [36] are:

$$\langle N \rangle = \mathcal{N}_m^2(|z|^2; a_r) \sum_{n=0}^{\infty} E_{n+m} \frac{|z|^{2n}}{|K_n^m(a_r)|^2}, \quad \langle N^2 \rangle = \mathcal{N}_m^2(|z|^2; a_r) \sum_{n=0}^{\infty} E_{n+m}^2 \frac{|z|^{2n}}{|K_n^m(a_r)|^2}. \quad (2.22)$$

Using (2.22), the pseudo-thermal expectation values of the operator N and of its square N^2 , given by $\langle N \rangle^{(m)} = \text{Tr}(\rho^{(m)} N)$ and $\langle N^2 \rangle^{(m)} = \text{Tr}(\rho^{(m)} N^2)$, respectively, allow to obtain the thermal intensity correlation function as follows:

$$(g^2)^{(m)} = \frac{\langle N^2 \rangle^{(m)} - \langle N \rangle^{(m)^2}}{(\langle N \rangle^{(m)})^2}. \quad (2.23)$$

Then, the thermal analogue of the Mandel parameter given by

$$Q^{(m)} = \langle N \rangle^{(m)} \left[(g^2)^{(m)} - 1 \right] \quad (2.24)$$

is deduced.

3 Pöschl-Teller potential

Consider the family of potentials

$$V_{l,l'}(x) = \begin{cases} \frac{1}{4a^2} \left[\frac{l(l-1)}{\sin^2 u(x)} - \frac{l'(l'-1)}{\sin^2 u(x)} \right] - \frac{(l+l')^2}{4a^2}, & u(x) = \frac{x}{2a}, \quad 0 < x < \pi a \\ \infty, & x \leq 0, x \geq \pi a \end{cases} \quad (3.1)$$

of continuously indexed parameters l, l' . This class of potentials called Pöschl-Teller potentials of first type (PT-I), intensively studied in [3, 9, 18, 22], is closely related to other classes of potentials, widely used in molecular physics, namely (i) the symmetric Pöschl-Teller potentials well ($l = l' \geq 1$), (ii) the Scarf potentials $\frac{1}{2} < l' \leq 1$ [33], (iii) the modified Pöschl-Teller potentials which can be obtained by replacing the trigonometric functions by their hyperbolic counterparts [13, 31], (iv) the Rosen-Morse potential which is the symmetric modified Pöschl-Teller potentials [32].

Let us define the corresponding Hamiltonian operator $H_{l,l'}$ with the action

$$H_{l,l'} \phi := \left(-\frac{d^2}{dx^2} + \frac{1}{4a^2} \left[\frac{l(l-1)}{\sin^2(x/2a)} - \frac{l'(l'-1)}{\sin^2(x/2a)} \right] - \frac{(l+l')^2}{4a^2} \right) \phi \quad \text{for } \phi \in \mathcal{D}_{H_{l,l'}} \quad (3.2)$$

in the suitable Hilbert space $\mathcal{H} = L^2((0, \pi a), dx)$. $\mathcal{D}_{H_{l,l'}}$ is the domain of definition of $H_{l,l'}$. We consider here the case where $l, l' \geq 3/2$, then the operator $H_{l,l'}$ is in the limit point case at both ends $x = 0, \pi a$, therefore it is essentially self-adjoint. In this case (see [3, 9] for more details) the Pöschl-Teller Hamiltonian can be defined as the self-adjoint operator $H_{l,l'}$ in $L^2([0, \pi a], dx)$ acting as in (3.2), on the dense domain

$$\mathcal{D}_{H_{l,l'}} = \left\{ \phi \in AC^2(0, \pi a) \mid \left(\frac{1}{4a^2} \left[\frac{l(l-1)}{\sin^2(x/2a)} - \frac{l'(l'-1)}{\sin^2(x/2a)} \right] - \frac{(l+l')^2}{4a^2} \right) \phi \in L^2([0, \pi a], dx), \right. \\ \left. \phi(0) = \phi(\pi a) = 0 \right\}. \quad (3.3)$$

with $AC^2(0, \pi a) = \{ \phi \in ac^2(0, \pi a) : \phi' \in \mathcal{H} \}$, where $ac^2(0, \pi a)$ denotes the set of absolutely continuous functions with absolutely continuous derivatives.

PT-I potentials are SUSY and fullfill the property of shape invariance [10]. Their superpotentials are:

$$W(x; l, l') = -\frac{1}{2a} [l \cot(u(x)) - l' \tan(u(x))]. \quad (3.4)$$

One can define the first differential operators A, A^\dagger that factorize the Hamiltonian operator in (3.2) as:

$$A := \frac{d}{dx} + W(x; l, l'), \quad A^\dagger := -\frac{d}{dx} + W(x; l, l') \quad (3.5)$$

with the domains:

$$\mathcal{D}(A) = \{\psi \in ac[0, \pi a], \quad (\psi' + W(x; l, l')\psi) \in \mathcal{H}\} \quad (3.6)$$

$$\mathcal{D}(A^\dagger) = \left\{ \phi \in ac[0, \pi a] \mid \exists \tilde{\phi} \in \mathcal{H} : [\psi(x)\phi(x)]_0^a = 0, \langle A\psi, \phi \rangle = \langle \psi, \tilde{\phi} \rangle, \forall \psi \in \mathcal{D}(A) \right\} \quad (3.7)$$

with $A^\dagger \phi = \tilde{\phi}$. The partner potentials $V_{1,2}$ satisfy the following shape invariance relation:

$$V_2(x, l, l') = V_1(x, l+1, l'+1) + \frac{1}{a^2}(l+l'+1). \quad (3.8)$$

The potential parameters $a_1 \equiv (l, l')$ and $a_2 \equiv (l+1, l'+1)$ are related as

$$a_2 = a_1 + 2, \quad (3.9)$$

while the remainder in the shape invariant condition (1.14) is $\mathcal{R}(a_1) = \frac{1}{a^2}(l+l'+1)$. Then the products in terms of the quantity $\mathcal{R}(a_s)$ in the numerator and denominator of the coefficient $K_n^m(a_r)$, see Eq. (2.5), can be read, respectively, as:

$$\prod_{k=m+1}^{n+m} \left(\sum_{s=k}^{n+m} \mathcal{R}(a_s) \right) = \lambda^{2n} \frac{\Gamma(n+1)\Gamma(2n+2m+2\rho)}{\Gamma(n+2m+2\rho)} \quad (3.10)$$

$$\prod_{k=1}^m \left(\sum_{s=k}^{n+m} \mathcal{R}(a_s) \right) = \lambda^{2m} \frac{\Gamma(n+m+1)\Gamma(n+2m+2\rho)}{\Gamma(n+1)\Gamma(n+m+2\rho)} \quad (3.11)$$

where we set $\lambda = \frac{1}{a}$ and $2\rho = l+l', \rho \geq 3/2$. The explicit form of the expansion coefficient $K_n^m(a_r)$ depends on the choice of the functional \mathcal{Z}_j .

3.1 First choice of the functional \mathcal{Z}_j

First we define the functional \mathcal{Z}_j as $\mathcal{Z}_j = e^{-i\alpha\mathcal{R}(a_1)}$, then we obtain

$$\prod_{k=m}^{n+m-1} \mathcal{Z}_{j+k} = e^{i\alpha E_n}, \quad E_n = \lambda^2 n(n+2\rho). \quad (3.12)$$

Inserting this relation and the results (3.10) and (3.11) in (2.5), we obtain the expansion coefficient as:

$$K_n^m(a_r) = \lambda^{n-m} \sqrt{\frac{\Gamma(n+1)^2 \Gamma(2n+2m+2\rho) \Gamma(n+m+2\rho)}{\Gamma(n+m+1)\Gamma(n+2m+2\rho)^2}} e^{i\alpha E_n}. \quad (3.13)$$

(i) *Normalization*

The normalization factor, in terms of the generalized hypergeometric functions ${}_3F_4$, can readily be deduced from (2.6) as:

$$\mathcal{N}_m(|z|^2; a_r) = \left[\frac{\lambda^{2m} \Gamma(m+1) \Gamma(2m+2\rho)}{\Gamma(m+2\rho)} {}_3F_4 \left(\begin{matrix} m+1, 2m+2\rho, 2m+2\rho \\ 1, m+\rho, m+2\rho, m+\rho+1/2 \end{matrix} ; \frac{|az|^2}{4} \right) \right]^{-1/2} \quad (3.14)$$

In terms of Meijer's G-function, we have:

$$\mathcal{N}_m(|z|^2; a_r) = \left[\frac{\lambda^{2m} \Gamma(m+\rho) \Gamma(m+\rho+\frac{1}{2})}{\Gamma(2m+2\rho)} G_{3,5}^{1,3} \left(-\frac{|az|^2}{4} \middle| \begin{matrix} -m, 1-2m-2\rho, 1-2m-2\rho \\ 0, 0, 1-m-\rho, 1-m-2\rho, 1/2-m-\rho \end{matrix} \right) \right]^{-1/2} \quad (3.15)$$

The explicit form of these PA-SIPCS are:

$$|z; a_r\rangle_m = \mathcal{N}_m(|z|^2; a_r) \lambda^m \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+m+1) \Gamma(n+2m+2\rho)^2}{\Gamma(n+m+2\rho) \Gamma(2n+2m+2\rho)}} \frac{(az)^n}{n!} |n+m\rangle \quad (3.16)$$

defined on the whole complex plane. For $m=0$, we recover the expansion coefficient and the normalization factor obtained in [2] for the generalized SIPCS:

$$\begin{aligned} K_n^0 &= \lambda^n \sqrt{\frac{\Gamma(n+1) \Gamma(2\rho+2n)}{\Gamma(2\rho+n)}} e^{i\alpha E_n} = h_n(a_r), \\ \mathcal{N}_0(|z|^2; a_r) &= \left[{}_1F_2 \left(2\rho; \rho, \rho+1/2; \frac{|az|^2}{4} \right) \right]^{-1/2} = \mathcal{N}(|z|^2; a_r). \end{aligned} \quad (3.17)$$

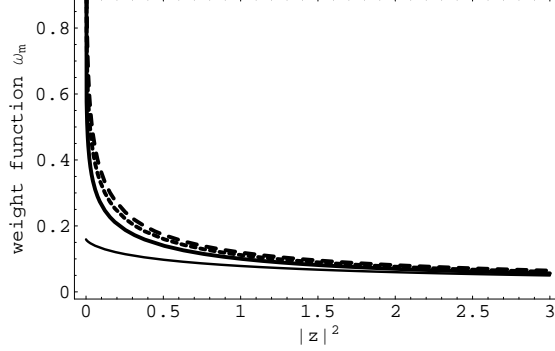


Figure 1: Plots of the weight function (3.19) of the PA-SIPCS (3.16) versus $|z|^2$ with the potential parameters $\rho=2, \lambda=1$, for different values of the photon added number m with $m=0$ (thin solid line), $m=1$ (solid line), $m=3$ (dot line), and $m=4$ (dashed line).

(ii) Non-orthogonality

The inner product of two different PA-SIPCS $|z; a_r\rangle_m$ and $|z'; a_r\rangle_{m'}$ follows from Eq (2.8):

$${}_m \langle z'; a_r | z; a_r \rangle_m = \chi(z', z, m, m', \rho) {}_3F_4 \left(\begin{matrix} m+1, 2m+2\rho, m+m'+2\rho \\ m-m'+1, m+2\rho, m+\rho, m+\rho+\frac{1}{2} \end{matrix} ; \frac{a^2 z'^* z}{4} \right)$$

where $\chi(z', z, m, m', \rho) = \mathcal{N}_{m'}(|z'|^2; a_r) \mathcal{N}_m(|z|^2; a_r) z'^{(m-m')} \lambda^{(m+m')} \frac{\Gamma(m+1) \Gamma(m+m'+2\rho)}{\Gamma(m-m'+1) \Gamma(m+2\rho)}$.

(iii) Overcompleteness

The non-negative weight function $\omega_m(|z|^2; a_r)$ is related to the function g_m satisfying (2.13):

$$\int_0^\infty dx x^{n+m} g_m(x; a_r) = \xi(x, n, m, \rho) \frac{\Gamma(n+1)^2 \Gamma(n+m+2\rho) \Gamma(n+m+\rho) \Gamma(n+m+\rho+\frac{1}{2})}{\Gamma(n+m+1) \Gamma(n+2m+2\rho)^2} \quad (3.18)$$

where x stands for $|z|^2$, $\xi(x, n, m, \rho) = \lambda^{2(n-m)} \frac{2^{(2n+2m+2\rho)}}{2\sqrt{\pi}}$ and $\omega_m = \frac{x^m g_m(x; a_r)}{\pi N_m^2(x; a_r)}$. After variable change $n+m \rightarrow s-1$ and using the Mellin inversion theorem in terms of Meijer's G-function (2.15), we deduce:

$$\omega_m(|z|^2; a_r) = \frac{1}{2\pi\sqrt{\pi}} \frac{|z|^{2m}}{\mathcal{N}_m(|z|^2; a_r)^2} \lambda^{-2(1+2m)} 2^{2(1-\rho)} G_{3,5}^{5,0} \left(\frac{|az|^2}{4} \middle| \begin{matrix} 0, -1+2\rho+m, -1+2\rho+m \\ -m, -m, 2\rho-1, -1+\rho, -1/2+\rho \end{matrix} \right). \quad (3.19)$$

The weight function (3.19) is positive for the parameter $\rho > 0$ as shown in Figure 1, where the curves are represented for $\rho = 2$ and for $m = 0, 1, 2, 3$. All the functions are positive for $x = |z|^2 \in \mathbb{R}_+$ and tend asymptotically to the measure of the conventional CS ($m = 0$). The measure has a singularity at $x = 0$ and tends to zero for $x \rightarrow \infty$.

(iv) *Thermal statistics*

Consider the normalized density operator expression

$$\rho^{(m)} = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\beta E_n} |n+m\rangle \langle n+m| \quad (3.20)$$

in which the exponent βE_n is re-cast as follows: $\beta E_n = \beta \lambda^2 [n^2 + 2n\rho] = An^2 - B_\rho n$ where $A = \beta \lambda^2$, $B_\rho = -2\beta \rho \lambda^2$. Then, the energy exponential can be expanded in the power series, (see for e.g., [30]) such that

$$e^{-\beta E_n} = e^{-An} \left[\sum_{k=0}^{\infty} \frac{(B_\rho)^k}{k!} n^{2k} \right] = \left\{ \sum_{k=0}^{\infty} \frac{(B_\rho)^k}{k!} \left(\frac{d}{dA} \right)^{2k} \right\} (e^{-A})^n = \exp \left[B_\rho \left(\frac{d}{dA} \right)^2 \right] (e^{-A})^n. \quad (3.21)$$

Thereby,

$$\rho^{(m)} = \frac{\exp \left[B_\rho \left(\frac{d}{dA} \right)^2 \right]}{Z} \sum_{n=0}^{\infty} (e^{-A})^n |n+m\rangle \langle n+m|. \quad (3.22)$$

From (3.16) and (3.22), we get, in terms of Meijer's G functions, the Q -distribution or Husimi distribution:

$$\begin{aligned} {}_m\langle z; m | \rho^{(m)} | z; m \rangle_m &= \frac{\Gamma(2m+2\rho)}{\Gamma(m+\rho)\Gamma(m+\rho+1/2)} \frac{\Gamma(\frac{1}{2})}{2^{2(m+\rho-1/2)}} \frac{\exp \left[B_\rho \left(\frac{d}{dA} \right)^2 \right]}{Z} \times \\ &\times \frac{G_{3,5}^{1,3} \left(-\frac{(a|z|^2)^2}{4} e^{-A} \middle| \begin{matrix} -m, 1-2m-2\rho, 1-2m-2\rho; \\ 0; 0, 1-m-\rho, 1-m-2\rho, 1/2-m-\rho \end{matrix} \right)}{G_{3,5}^{5,0} \left(-\frac{(a|z|^2)^2}{4} \middle| \begin{matrix} -m, 1-2m-2\rho, 1-2m-2\rho; \\ 0; 0, 1-m-\rho, 1-m-2\rho, 1/2-m-\rho \end{matrix} \right)}. \end{aligned} \quad (3.23)$$

The angular integration achieved, taking $x = |z|^2$, the condition (2.18) supplies

$$\begin{aligned} \text{Tr} \rho^{(m)} &= \frac{\exp \left[B_\rho \left(\frac{d}{dA} \right)^2 \right]}{Z} \frac{2^{2[1-2\rho-m]}}{\lambda^{2(1+m)}} \int_0^\infty dx x^m G_{3,5}^{1,3} \left(-\frac{|a|^2}{4} x e^{-A} \middle| \begin{matrix} -m, 1-2m-2\rho, 1-2m-2\rho; \\ 0; 0, 1-m-\rho, 1-m-2\rho, 1/2-m-\rho \end{matrix} \right) \times \\ &\times G_{3,5}^{5,0} \left(\frac{|a|^2}{4} x \middle| \begin{matrix} ; 0, -1+2m+\rho, -1+2\rho+m \\ -m, -m, 2\rho-1, -1+\rho, -1/2+\rho; \end{matrix} \right). \end{aligned} \quad (3.24)$$

Then, the integral of Meijer's G-function product properties provides the partition function

$$Z = \frac{2^{4(1-\rho)}}{(\lambda|a|)^{2(1+m)}} \exp \left[B_\rho \left(\frac{d}{dA} \right)^2 \right] \sum_{n=0}^{\infty} (e^{-A})^n. \quad (3.25)$$

From (2.19), using the result $\langle n+m | \rho^{(m)} | n+m \rangle = \frac{1}{Z} \exp \left[B_\rho \left(\frac{d}{dA} \right)^2 \right] (e^{-A})^n$ and setting $\bar{n}_A = (e^A - 1)^{-1}$, we get the following integration equality

$$\begin{aligned} \frac{1}{\bar{n}_A + 1} \left(\frac{\bar{n}_A}{\bar{n}_A + 1} \right)^n &\frac{\Gamma(n+1)^2 \Gamma(n+m+2\rho) \Gamma(2n+2m+2\rho)}{\Gamma(n+m+1) \Gamma(n+2m+2\rho)^2} \frac{\sqrt{\pi} \lambda^{4(1+m)}}{2^{5-6\rho} |a|^{2(n-m-1)}} = \int_0^\infty dx x^{n+m} P(x) \times \\ &G_{3,5}^{5,0} \left(\frac{|a|^2}{4} x \middle| \begin{matrix} ; 0, -1+2m+\rho, -1+2\rho+m \\ -m, -m, 2\rho-1, -1+\rho, -1/2+\rho; \end{matrix} \right). \end{aligned}$$

After performing the exponent change $n+m = s-1$ in order to get the Stieltjes moment problem, we arrive at the P -function as

$$P(|z|^2) = \frac{1}{\bar{n}_A} \left(\frac{\bar{n}_A + 1}{\bar{n}_A} \right)^m \frac{\lambda^2 |a|^{2(m+1)}}{2^{4(1-\rho)}} \frac{G_{3,5}^{5,0} \left(\frac{\bar{n}_A + 1}{\bar{n}_A} \frac{|az|^2}{4} \middle| \begin{matrix} ; 0, -1+2\rho+m, -1+2\rho+m \\ -m, -m, 2\rho-1, -1+\rho, -1/2+\rho; \end{matrix} \right)}{G_{3,5}^{5,0} \left(\frac{|az|^2}{4} \middle| \begin{matrix} ; 0, -1+2\rho+m, -1+2\rho+m \\ -m, -m, 2\rho-1, -1+\rho, -1/2+\rho; \end{matrix} \right)} \quad (3.26)$$

which obeys the normalization to unity condition (2.20).

Then, the diagonal representation of the normalized density operator in terms of the PA-SIPCS projector (2.19) takes the form

$$\rho^{(m)} = \frac{1}{\bar{n}_A} \left(\frac{\bar{n}_A + 1}{\bar{n}_A} \right)^m \frac{\lambda^2 |a|^{2(m+1)}}{2^{4(1-\rho)}} \int_{\mathbb{C}} d^2 z \omega_m(|z|^2; a_r) |z; a_r\rangle_m \mathfrak{S}_{3,5}^{5,0}(|z|^2; \bar{n}_A)_m \langle z; a_r| \quad (3.27)$$

with $\mathfrak{S}_{3,5}^{5,0}(|z|^2, \bar{n}_A)$ - the Meijer's G-functions quotient given in (3.26). Using the relations (3.26), (3.27), and the definition (2.21), the pseudo-thermal expectation values of the operator N and its square are given by

$$\begin{aligned} \langle N \rangle^{(m)} &= \left(\frac{|a|^2}{4} \right)^{m+1} \frac{1}{\lambda^{2(m-1)}} \frac{m(m+2\rho)}{(m+1)(m+1+2\rho)} \\ &\times \left[1 + \left(\frac{1}{m+1} + \frac{1}{m+1+2\rho} \right) \bar{n} + \frac{1}{(m+1)(m+1+2\rho)} \left(\frac{\bar{n}}{1-e^{-\beta}} + \bar{n}^2 \right) \right] \end{aligned} \quad (3.28)$$

$$\begin{aligned} \langle N^2 \rangle^{(m)} &= \left(\frac{|a|^2}{4} \right)^{m+1} \frac{1}{\lambda^{2(m-2)}} \left[\frac{m(m+2\rho)}{(m+1)(m+1+2\rho)} \right]^2 \times \left\{ 1 + 2 \left(\frac{1}{m+1} + \frac{1}{m+1+2\rho} \right) \bar{n} + \right. \\ &+ \left(\frac{1}{(m+1)^2} + \frac{1}{(m+1+2\rho)^2} + \frac{4}{(m+1)(m+1+2\rho)} \right) \left(\frac{\bar{n}}{1-e^{-\beta}} + \bar{n}^2 \right) + \\ &+ 2 \left(\frac{1}{(m+1)^2(m+1+2\rho)} + \frac{1}{(m+1)(m+1+2\rho)^2} \right) \left[\frac{\bar{n}}{(1-e^{-\beta})^2} + \frac{4\bar{n}^2}{1-e^{-\beta}} + \bar{n}^3 \right] + \\ &\left. + \frac{1}{(m+1)^2(m+1+2\rho)^2} \left[\frac{\bar{n}}{(1-e^{-\beta})^3} + \frac{11\bar{n}^2}{(1-e^{-\beta})^2} + \frac{11\bar{n}^3}{1-e^{-\beta}} + \bar{n}^4 \right] \right\} \end{aligned} \quad (3.29)$$

where $\bar{n} = (e^{-\beta} - 1)^{-1}$. Thereby,

$$\begin{aligned} (g^2)^{(m)} &= 1 + \left\{ \left(\frac{1}{(m+1)} + \frac{1}{(m+1+2\rho)} \right)^2 \frac{\bar{n}}{(1-e^{-\beta})} + \right. \\ &\left(\frac{1}{(m+1)^2(m+1+2\rho)} + \frac{1}{(m+1)(m+1+2\rho)^2} \right) \left[\frac{2\bar{n}}{(1-e^{-\beta})^2} + \frac{6\bar{n}^2}{1-e^{-\beta}} \right] + \\ &\frac{1}{(m+1)^2(m+1+2\rho)^2} \left[\frac{\bar{n}}{(1-e^{-\beta})^3} + \frac{10\bar{n}^2}{(1-e^{-\beta})^2} + \frac{9\bar{n}^3}{1-e^{-\beta}} \right] \Bigg\} \\ &\times \frac{1}{\left(\frac{|a|^2}{4} \right)^{-(m+1)} \lambda^{2m} (\langle N \rangle^{(m)})^2} - \frac{1}{\left(\frac{|a|^2}{4} \right)^{-\frac{m+1}{2}} \lambda^m \langle N \rangle^{(m)}}. \end{aligned} \quad (3.30)$$

Then, the thermal analogue of the Mandel parameter is given by

$$\begin{aligned} Q^{(m)} &= \left(\frac{|a|^2}{4} \right)^{-\frac{m+1}{2}} \lambda^m \langle N \rangle^{(m)} \left[(g^2)^{(m)} - 1 \right] \\ &= \left\{ \left(\frac{1}{(m+1)} + \frac{1}{(m+1+2\rho)} \right)^2 \frac{\bar{n}}{(1-e^{-\beta})} + \right. \\ &\left(\frac{1}{(m+1)^2(m+1+2\rho)} + \frac{1}{(m+1)(m+1+2\rho)^2} \right) \left[\frac{2\bar{n}}{(1-e^{-\beta})^2} + \frac{6\bar{n}^2}{1-e^{-\beta}} \right] + \\ &\frac{1}{(m+1)^2(m+1+2\rho)^2} \left[\frac{\bar{n}}{(1-e^{-\beta})^3} + \frac{10\bar{n}^2}{(1-e^{-\beta})^2} + \frac{9\bar{n}^3}{1-e^{-\beta}} \right] \Bigg\} \\ &\times \frac{1}{\left(\frac{|a|^2}{4} \right)^{-\frac{m+1}{2}} \lambda^m \langle N \rangle^{(m)}} - 1. \end{aligned} \quad (3.31)$$

3.2 Second choice of the functional \mathcal{Z}_j

We now take $\mathcal{Z}_j = \sqrt{g(a_1; \kappa, \kappa)g(a_1; \kappa, 0)} e^{-i\alpha\mathcal{R}(a_1)}$ with κ a real constant and where we use the auxiliary function [2] $g(a_j; c, d) = ca_j + d$, c and d being real constants. From the potential parameter relations (3.9)

we obtain:

$$\prod_{k=m}^{n+m-1} g(a_{j+k}; c, d) = 2c^n \frac{\Gamma(n+m+\frac{a_1}{2}+j-1+d/2c)}{\Gamma(m+\frac{a_1}{2}+j-1+d/2c)}. \quad (3.32)$$

Setting $a_1 = 2\rho$, we have:

$$\prod_{k=m}^{n+m-1} \mathcal{Z}_{j+k} = \left[\kappa^{2n} \frac{\Gamma(2n+2m+2\rho)}{\Gamma(2m+2\rho)} \right]^{\frac{1}{2}} e^{-i\alpha E_n} \quad (3.33)$$

with the eigen-energy E_n given by (3.12). Inserting Eqs. (3.33), (3.10) and (3.11) in the expansion coefficient (2.5), we obtain

$$K_n^m(a_r) = \left[\frac{1}{\kappa^{2m}} \frac{\Gamma(n+1)^2 \Gamma(n+m+\nu+1) \Gamma(2m+\nu+1)}{\Gamma(n+m+1) \Gamma(n+2m+\nu+1)^2} \right]^{\frac{1}{2}} e^{i\alpha E_n}, \quad (3.34)$$

where we assume $\lambda = \frac{1}{a} = \kappa$ and $\rho = \frac{\nu}{2} + \frac{1}{2}, \nu \geq 1$ in (3.10) and (3.11). For $m = 0$, we recover the coefficient h_n in [2]:

$$K_n^0(a_r) = \left[\frac{\Gamma(n+1) \Gamma(\nu+1)}{\Gamma(n+\nu+1)} \right]^{\frac{1}{2}} e^{i\alpha E_n} = h_n(a_r). \quad (3.35)$$

(i) *Normalization*

The normalization factor in terms of hypergeometric and Meijer's G-functions is

$$\mathcal{N}_m(|z|^2; a_r) = \kappa^{2m} \Gamma(m+1) \frac{\Gamma(2m+\nu+1)}{\Gamma(m+\nu+1)} \left[{}_3F_2 \left(\begin{matrix} m+1, 2m+\nu+1, 2m+\nu+1 \\ 1, m+\nu+1 \end{matrix} ; |z|^2 \right) \right]^{-\frac{1}{2}} \quad (3.36)$$

$$\mathcal{N}_m(|z|^2; a_r) = \left[\frac{\kappa^{2m}}{\Gamma(2m+\nu+1)} G_{3,3}^{1,3} \left(-|z|^2 \left| \begin{matrix} -m, -2m-\nu, -2m-\nu \\ 0, 0, -m-\nu \end{matrix} \right. \right) \right]^{-\frac{1}{2}}. \quad (3.37)$$

The explicit form of the PA-SIPCS, defined for $|z| < 1$, is provided by:

$$|z; a_r\rangle_m = \mathcal{N}_m(|z|^2; a_r) \sum_{n=0}^{\infty} \sqrt{\kappa^{2m} \frac{\Gamma(n+m+1) \Gamma(n+2m+\nu+1)^2}{\Gamma(n+1)^2 \Gamma(n+m+\nu+1) \Gamma(2m+\nu+1)}} z^n e^{-i\alpha E_n} |n+m\rangle. \quad (3.38)$$

For $m = 0$, we recover the normalization factor

$$\mathcal{N}_0(|z|^2; a_r) = {}_1F_0(\nu+1; -; |z|^2)^{-\frac{1}{2}} = (1-|z|^2)^{-1/2-\nu/2} = \mathcal{N}(|z|^2; a_r) \quad (3.39)$$

obtained in [2]. For $m = 0$, the PA-SIPCS is reduced to the SIPCS

$$|z; a_r\rangle = (1-|z|^2)^{(\nu+1)/2} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+\nu+1)}{\Gamma(\nu+1) \Gamma(n+1)}} e^{-i\alpha E_n} |\Psi_n\rangle \quad (3.40)$$

obtained in [2] and in [22] as CS of Klauder-Perelomov's type for the PT-I.

(ii) *Non-orthogonality*

The inner product of two different PA-SIPCS $|z; a_r\rangle_m$ and $|z'; a_r\rangle_{m'}$ is given by:

$${}_{m'} \langle z'; a_r | z; a_r \rangle_m = \chi(z', z, m, m', \nu) {}_3F_2 \left(\begin{matrix} m+1, m+m'+\nu+1, 2m+\nu+1 \\ m-m'+1, m+\nu+1 \end{matrix} ; z'^* z \right)$$

where

$$\begin{aligned} \chi(z', z, m, m', \nu) &= \mathcal{N}_{m'}(|z'|^2; a_r) \mathcal{N}_m(|z|^2; a_r) \frac{z'^*(m-m') \kappa^{(m+m')}}{\sqrt{\Gamma(2m+\nu+1) \Gamma(2m'+\nu+1)}} \times \\ &\times \frac{\Gamma(m+1) \Gamma(m+m'+\nu+1) \Gamma(2m+\nu+1)}{\Gamma(m-m'+1) \Gamma(m+\nu+1)} e^{i\alpha(E_n - E_{n+m-m'})}. \end{aligned}$$

(iii) *Overcompleteness*

Following the steps of section 2.3, we obtain the weight-function of the PA-SIPCS (3.38) as

$$\omega_m(|z|^2; a_r) = \frac{1}{\pi} G_{3,3}^{1,3} \left(-|z|^2 \left| \begin{matrix} -m, -2m-\nu, -2m-\nu \\ 0, 0, -m-\nu \end{matrix} \right. \right) G_{3,3}^{3,0} \left(|z|^2 \left| \begin{matrix} m, 2m+\nu, 2m+\nu \\ 0, 0, m+\nu \end{matrix} \right. \right). \quad (3.41)$$

We recover, for $m = 0$, the result:

$$\omega_0(|z|^2; a_r) = \frac{\Gamma(\nu+1)}{\pi} {}_1F_0(\nu+1; -; |z|^2) G_{1,1}^{1,0} \left(|z|^2 \left| \begin{matrix} \nu \\ 0 \end{matrix} \right. \right) = \frac{\nu}{\pi} (1 - |z|^2)^{-2} \quad (3.42)$$

obtained in [2] for the corresponding ordinary SIPCS.

(iv) *Thermal statistics*

Since the eigen-energy E_n (3.12) is the same as previously, we start by maintaining the relations (3.20)-(3.22). From (3.38) and (3.22), we get, in terms of Meijer's G functions, the Q -distribution or Husimi distribution

$${}_m\langle z; m | \rho^{(m)} | z; m \rangle_m = \frac{\exp \left[B_\rho \left(\frac{d}{dA} \right)^2 \right]}{Z} \frac{G_{3,3}^{1,3} \left(-|z|^2 e^{-A} \left| \begin{matrix} -m, -2m-\nu, -2m-\nu \\ 0, 0, -m-\nu \end{matrix} \right. \right)}{G_{3,3}^{1,3} \left(-|z|^2 \left| \begin{matrix} -m, -2m-\nu, -2m-\nu \\ 0, 0, -m-\nu \end{matrix} \right. \right)}. \quad (3.43)$$

The angular integration achieved, taking $x = |z|^2$, the condition (2.18) supplies

$$\text{Tr} \rho^{(m)} = \frac{\exp \left[B_\rho \left(\frac{d}{dA} \right)^2 \right]}{Z} \int_0^\infty dx x^m G_{3,3}^{3,0} \left(|z|^2 \left| \begin{matrix} m, 2m+\nu, 2m+\nu \\ 0, 0, m+\nu \end{matrix} \right. \right) \times \\ G_{3,3}^{1,3} \left(-|z|^2 e^{-A} \left| \begin{matrix} -m, -2m-\nu, -2m-\nu \\ 0, 0, -m-\nu \end{matrix} \right. \right). \quad (3.44)$$

Then, the integral of Meijer's G-function product properties provides the partition function expression $Z = \exp \left[B_\rho \left(\frac{d}{dA} \right)^2 \right] \sum_{n=0}^\infty (e^{-A})^n$. From (2.19), taking $\bar{n}_A = (e^A - 1)^{-1}$, we get the following integration equality

$$\frac{1}{\bar{n}_A + 1} \left(\frac{\bar{n}_A}{\bar{n}_A + 1} \right)^n \frac{\Gamma(n+1)^2 \Gamma(n+m+\nu+1)}{\Gamma(n+m+1) \Gamma(n+2m+\nu+1)^2} = \int_0^\infty dx x^n P(x) G_{3,3}^{3,0} \left(|z|^2 \left| \begin{matrix} m, 2m+\nu, 2m+\nu \\ 0, 0, m+\nu \end{matrix} \right. \right).$$

Finally, we arrive at the P -function:

$$P(|z|^2) = \frac{1}{\bar{n}_A} \frac{G_{3,3}^{3,0} \left(\frac{\bar{n}_A+1}{\bar{n}_A} |z|^2 \left| \begin{matrix} m, 2m+\nu, 2m+\nu \\ 0, 0, m+\nu \end{matrix} \right. \right)}{G_{3,3}^{3,0} \left(|z|^2 \left| \begin{matrix} m, 2m+\nu, 2m+\nu \\ 0, 0, m+\nu \end{matrix} \right. \right)} \quad (3.45)$$

which obeys the normalization to unity condition (2.20).

Then, the diagonal representation of the normalized density operator in terms of the PA-SIPCS projector (2.19) takes the form

$$\rho^{(m)} = \frac{1}{\bar{n}_A} \int_{\mathbb{T}} d^2 z \omega_m(|z|^2; a_r) |z; a_r\rangle_m \mathfrak{S}_{3,3}^{3,0}(|z|^2; \bar{n}_A) {}_m\langle z; a_r| \quad (3.46)$$

with $\mathfrak{S}_{3,3}^{3,0}(|z|^2, \bar{n}_A)$ - the Meijer's G-functions quotient given in (3.45). Using the relations (3.45), (3.46), and the definition (2.21), the pseudo-thermal expectation values of the operator N and its square are given by

$$\langle N \rangle^{(m)} = \kappa^2 m(m+\nu+1) \left[1 + \left(\frac{1}{m+1} + \frac{1}{m+\nu+2} \right) \bar{n} + \frac{1}{(m+1)(m+\nu+2)} \left(\frac{\bar{n}}{1-e^{-\beta}} + \bar{n}^2 \right) \right] \quad (3.47)$$

$$\begin{aligned}
\langle N^2 \rangle^{(m)} &= \kappa^4 m^2 (m + \nu + 1)^2 \left\{ 1 + 2 \left(\frac{1}{m+1} + \frac{1}{m+\nu+2} \right) \bar{n} + \right. \\
&\quad \left(\frac{1}{(m+1)^2} + \frac{1}{(m+\nu+2)^2} + \frac{4}{(m+1)(m+\nu+2)} \right) \left(\frac{\bar{n}}{1-e^{-\beta}} + \bar{n}^2 \right) + \\
&\quad 2 \left(\frac{1}{(m+1)^2(m+\nu+2)} + \frac{1}{(m+1)(m+\nu+2)^2} \right) \left(\frac{\bar{n}}{(1-e^{-\beta})^2} + \frac{4\bar{n}^2}{1-e^{-\beta}} + \bar{n}^3 \right) + \\
&\quad \left. \frac{1}{(m+1)^2(m+\nu+2)^2} \left[\frac{\bar{n}}{(1-e^{-\beta})^3} + \frac{11\bar{n}^2}{(1-e^{-\beta})^2} + \frac{11\bar{n}^3}{1-e^{-\beta}} + \bar{n}^4 \right] \right\}. \quad (3.48)
\end{aligned}$$

Thereby,

$$\begin{aligned}
(g^2)^{(m)} &= 1 + \left\{ \left(\frac{1}{m+1} + \frac{1}{m+\nu+2} \right)^2 \frac{\bar{n}}{(1-e^{-\beta})} + \left(\frac{1}{(m+1)^2(m+\nu+2)} \right. \right. \\
&\quad \times \left. \left. + \frac{1}{(m+1)(m+\nu+2)^2} \right) \left[\frac{2\bar{n}}{(1-e^{-\beta})^2} + \frac{6\bar{n}^2}{1-e^{-\beta}} \right] + \frac{1}{(m+1)^2(m+\nu+2)^2} \right. \\
&\quad \left. \left[\frac{\bar{n}}{(1-e^{-\beta})^3} + \frac{10\bar{n}^2}{(1-e^{-\beta})^2} + \frac{9\bar{n}^3}{1-e^{-\beta}} \right] \right\} \frac{1}{(\langle N \rangle^{(m)})^2} - \frac{1}{\langle N \rangle^{(m)}}. \quad (3.49)
\end{aligned}$$

Then, the thermal analogue of the Mandel parameter is given by

$$\begin{aligned}
Q^{(m)} &= \langle N \rangle^{(m)} \left[(g^2)^{(m)} - 1 \right] = \left\{ \left(\frac{1}{m+1} + \frac{1}{m+\nu+2} \right)^2 \frac{\bar{n}}{(1-e^{-\beta})} + \right. \\
&\quad \left(\frac{1}{(m+1)^2(m+\nu+2)} + \frac{1}{(m+1)(m+\nu+2)^2} \right) \left(\frac{2\bar{n}}{(1-e^{-\beta})^2} + \frac{6\bar{n}^2}{1-e^{-\beta}} \right) + \\
&\quad \left. \frac{1}{(m+1)^2(m+\nu+2)^2} \left[\frac{\bar{n}}{(1-e^{-\beta})^3} + \frac{10\bar{n}^2}{(1-e^{-\beta})^2} + \frac{9\bar{n}^3}{1-e^{-\beta}} \right] \right\} \frac{1}{\langle N \rangle^{(m)}} - 1. \quad (3.50)
\end{aligned}$$

4 Concluding remarks

In this contribution paper, we have shown the use of the shape invariant potential method to construct generalized coherent states for photon-added particle systems under Pöschl-Teller potentials. These states have been fully characterized and discussed from both mathematics and physics points of view. This algebro-operator method can be exploited to investigate a larger class of solvable potentials.

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